Simulation of a Random Differential Equation

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The problem considered is that of a second-order differential equation with stochastic parameters. Time-dependent random coefficients may assume both positive and negative values when modeling parametric excitations. As there is no exact solution in the case of colored noise random coefficients, one must use approximate techniques. We presently investigate an iterative procedure. The quality of this approximation to the exact solution is verified with a Monte Carlo simulation. The particular example considered is that of a harmonic oscillator with a time-dependent random stiffness and excited by an external random forcing function. After a brief review of the iterative method, and an outline of the design of the Monte Carlo simulation, an extensive parametric study is presented to establish ranges of parameter values for which the approximation is valid. This comparison study leads to a design criterion for the mathematical modeling of structures with parametric uncertainties. We are interested in using information from studies such as this to understand the behavior of large-scale structures.

Introduction

GIVEN a statistical description of the random parameters and of the forcing function, we wish to obtain the equivalent statistical description of the response. The mean value, autocorrelation, and spectral density of the response are the particular quantities presently derived. The mathematical theory presented here can be applied to two types of physical problems. The first type of problem is that the material properties of a structure are time-dependent random processes. For instance, temperature fluctuations that are known to affect the stiffness of composite structures may be of a random nature. The second class of problems is that of parametric random vibrations. The helicopter rotor blade problem, axially loaded panels, and sloshing of water in tanks are among a few of the applications listed in the excellent review by Ibrahim.[1] In these examples, the governing equations reduce to the form studied in this paper.

Although the analysis is performed here for the generic example of a harmonic oscillator with random stiffness, the applicability of the proposed method to structural analysis is a goal. Therefore, a design criterion delineating regions of applicability is established.

Theoretical Background

The present method is known under the name Born expansion in physics[2]; when only one term is considered, it is called a Born approximation. This method has been applied to several fields and, in particular, that of wave propagation in random media[2] and may be considered a perturbation method.

Results of previous studies[3-6] are recalled next. The nondimensional form of the governing equation is

$$\ddot{x}(t)+2\zeta\dot{x}(t)+[1+K(t)]x(t)=F(t) \tag{1}$$

where a dot denotes differentiation with respect to time, $F(t)$ and $K(t)$ are independent, centered random processes with Gaussian distributions, ζ is the damping ratio, and rest initial conditions are assumed. The solution to Eq. (1) is sought under the form

$$x(t)=\sum_{i=0}^{\infty} x_i(t) \tag{2}$$

where $x_0(t)$ is the response of the deterministic system to $F(t)$

$$x_0(t)=L_t^{-1}[F(t)] \tag{3}$$

and L_t is the deterministic component of the operator in Eq. (1). The general term of the series (2) is

$$x_i(t)=-L_t^{-1}[K(t)x_{i-1}(t)] \tag{4}$$

In particular, for $i=1$,

$$x_1(t)=-L_t^{-1}[K(t)x_0(t)] \tag{5}$$

The exact solution $x(t)$ is approximated by $\tilde{x}(t)$, where

$$\tilde{x}(t)=x_0(t)+x_1(t) \tag{6}$$

The corresponding spectral density is

$$\tilde{S}_{xx}(\omega)=S_{x_0x_0}(\omega)+S_{x_1x_1}(\omega) \tag{7}$$

where

$$S_{x_0x_0}(\omega)=S_{FF}(\omega)|H(i\omega)|^2 \tag{8}$$

and

$$S_{x_1x_1}(\omega)=\frac{1}{2\pi}[S_{x_1x_1}(\omega)*S_{x_0x_0}(\omega)]|H(i\omega)|^2 \tag{9}$$

A star denotes convolution; $S_{KK}(\omega)$, $S_{FF}(\omega)$ are the spectral densities of $K(t)$ and $F(t)$; $|H(i\omega)|^2$ is the deterministic system's transfer function. The cross terms $S_{x_0x_1}(\omega)$ $S_{x_1x_0}(\omega)$ are equal to zero because they contain an odd moment of $K(t)$, which is, without loss of generality, equal to zero.

This approximation will be referred to as the "one-term iterative method" in the following discussion. It was found that computations in the time domain were too complicated to be carried through analytically, but in the frequency domain the closed form expression for $S_{x_1x_1}(\omega)$ is available with the help of the symbolic algebra manipulation code MACSYMA.

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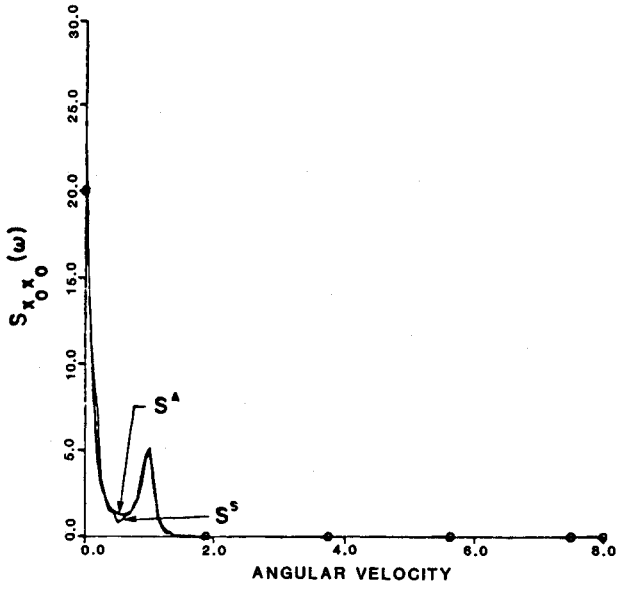

Fig. 1 Deterministic system, $\alpha = 0.1$ (S^A: analytic, S^S: simulated).

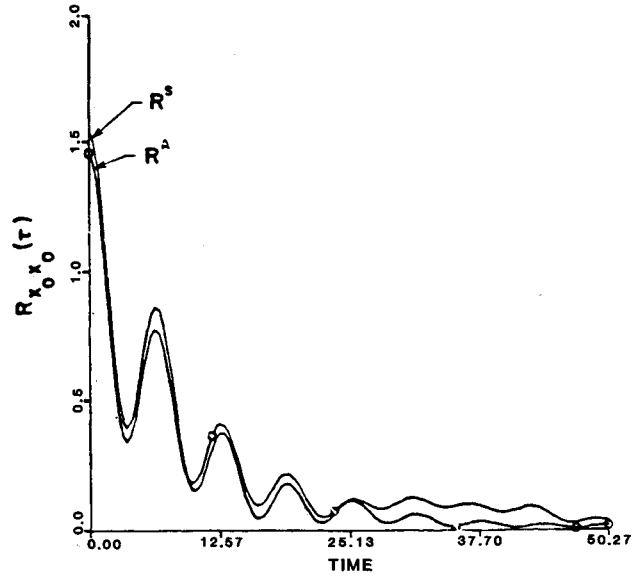

Fig. 2 Deterministic system, $\alpha = 0.1$ (R^A: analytic, R^S: simulated).

In Ref. 6 the special case where $K(t)$ is a white noise process was studied. The general term of the series of the spectral density is available and the infinite summation yields the exact solution obtained directly by solving the second-order moment equation. Backed by this encouraging result, it is expected that the proposed approximation replicates the simulation results for the colored noise case with sufficient accuracy.

Stability

Since $K(t)$ may assume both negative and positive values in this model, the system may be unstable in some cases. By instability, it is meant that the autocorrelation of $x(t)$ grows in an unbounded manner.

The approximation in Eq. (2) is composed of two terms: $x_0(t)$ and $x_1(t)$. Both of them can be interpreted as the response of the deterministic system to external random forcing functions: $x_0(t)$ is the response to $F(t)$ and $x_1(t)$ is the response to an effective force $K(t)x_0(t)$ that corresponds to a first cycle of iterations. Their respective autocorrelation functions decay exponentially with time and satisfy stability requirements. It is noted that although $x(t)$ may be unstable, $x_0(t)+x_1(t)$ is always stable because the two terms' respective autocorrelation functions are exponentially decaying in time.

In order to perform meaningful comparisons, the system's parameters should be such that $x(t)$ obtained by simulation is also stable. The most important parameters are the damping of the system ζ and the variance of $K(t)$, σ_K^2. Additional parameters that play a role in the stability are the decay constants of the autocorrelations that are assumed to be exponentially decaying with time: $R_{KK}(\tau)=\sigma_K^2 e^{-\beta|\tau|}$, $R_{FF}(\tau)=\sigma_F^2 e^{-\alpha|\tau|}$. These are β for $K(t)$ and α for $F(t)$. It is expected that for relatively large values of ζ, more excursions of $K(t)$ in the negative region are permissible without destabilizing the system. A system where $K(t)$ takes only positive values is always stable, whereas on the other extreme, a system without damping is always unstable.

In Ref. 6 the special case of white noise $K(t)$ of intensity S_K is investigated, and it is found that the absence of a statistical correlation between $K(t)$ and $x(t)$ made a closed form solution possible. In particular, the stability limit was $S_K < 4\zeta$. The equivalent expression for colored noise is unavailable to the best of our knowledge; we thus resort to numerical procedures to approximate this boundary.

Monte Carlo Simulation

The random processes $F(t)$ and $K(t)$ are replaced in Eq. (1) by specific values F_i and K_i that are distributed according to Gaussian distributions and are exponentially correlated. Equation (1) becomes deterministic and may be solved using standard methods for x_i. By repeating this procedure a certain number of times, one can estimate the autocorrelation of the process $x(t)$ from this large number of observations. The first step is thus to generate random numbers with the desired statistical properties. Next, the deterministic governing equation is solved a number of times. Finally, the autocorrelation is estimated from this set of observations. In practice, to minimize memory requirements, these steps are carried through simultaneously.

Generation of Colored Noise

Given a white noise process, one can generate a colored noise process with an exponentially decaying autocorrelation function by passing it through a first-order filter. That is, the colored noise is considered to be the solution of a first-order differential equation excited by a white noise forcing function. The filter equation is[7]

$$\dot{K}(t)+\beta K(t)=\sqrt{2\beta}\sigma_K^2 W(t) \tag{10}$$

The spectral density of $K(t)$ is

$$S_{KK}(\omega)=\sigma_K^2\frac{2\beta}{\beta^2+\omega^2} \tag{11}$$

and the autocorrelation is

$$R_{KK}(\tau)=\sigma_K^2 e^{-\beta|\tau|} \tag{12}$$

The corresponding discrete algorithm is

$$K_{i+1}=K_i e^{-\beta\Delta t}+cW_i, \qquad c=\sigma_K^2\sqrt{1-e^{-\beta\Delta t}} \tag{13}$$

where Δt is the sampling interval.

Estimation of the Autocorrelation and Spectral Density

The stochastic equation is treated as a deterministic equation for each time interval. Equation (1) becomes for $t_i \le t \le t_{i+1}$

$$\ddot{x}(t)+2\zeta\dot{x}(t)+(1+K_i)x(t)=F_i \tag{14}$$

with initial conditions $x_{0i}=x(t_i)$, $\dot{x}_{0i}=\dot{x}(t_i)$. The solution is obtained numerically with a three-point Runge-Kutta scheme.[8]

With M denoting the number of such Monte Carlo runs, the autocorrelation of $x(t)$ is estimated from the sample $\{x_1, x_2, \ldots, x_M\}$. If we assume that the process is ergodic, the autocorrelation is obtained by averaging the sample over time. Using the definition in Ref. 9

$$R(\tau) = \lim_{A \to \infty} \frac{1}{A} \int_0^A x(t)x(t+\tau)\, dt \tag{15}$$

The autocorrelation corresponding to the discrete process is[9]

$$R_k = \frac{1}{M} \sum_{j=1}^{M-k} x_j x_{j+k} \tag{16}$$

A fast Fourier transform is used to obtain the corresponding spectral density for comparison purposes with the one-term approximation.

Monte Carlo Design Parameters

Several parameters related to the simulation such as Δt, $\Delta\omega$, the sampling intervals, N, the number of points of the autocorrelation function; and M, the number of runs, are now discussed. Some of the decisions regarding their numerical values are made using physical reasoning, whereas others are made by trial and error with the help of preliminary runs for cases where exact analytic solutions are available. It was found that $\Delta t = \pi/8$ offers a good resolution for the curves, as well as avoids aliasing when performing transforms. The choice $\Delta\omega = \pi/N\Delta t$ ensures sampling consistency between curves that are obtained directly and those obtained by Fourier transformations. A value of $N = 128$ satisfies the requirements that the functions have practically decayed to zero after truncation and that it be a power of 2 for the fast Fourier algorithm. The number of runs $M = 80(N+1)$ is found to provide satisfactory fits between simulation and test cases.

Numerical Results

The Monte Carlo simulation results are compared at first with test cases where exact expressions are available. This allows for an appropriate choice of the Monte Carlo design parameters Δt, $\Delta\omega$, N, M just discussed. In particular, the case in which no randomness is present in the system will be of use later on to estimate the effect of parametric uncertainty on the deterministic system response. Next, the simulation is compared with the one-term iterative method for various decay constants and variances of the stiffness and forcing function. All curves are normalized with respect to the variance of the force. Tables of numerical values of the variance of the response and of spectral peaks at $\omega = 1$ are also compiled for comparison.

Deterministic System

The solution to the system without damping was denoted $x_0(t)$ in Eq. (3). The spectral density $S_{x_0x_0}(\omega)$ is given in Eq. (8) with $H(i\omega) = 1/(1-\omega^2+2i\omega)$ and $S_{FF}(\omega) = 2\alpha/(\alpha^2+\omega^2)$. The assumed damping ratio is $\zeta = 0.1$ and two values of α are considered (1.0, 0.1). Figure 1 shows comparisons between the simulation and the analytical spectral densities for $\alpha = 0.1$; here, $S^A = S_{x_0x_0}(\omega)$. The corresponding autocorrelation functions appear in Fig. 2. Figures 3 and 4 correspond to $\alpha = 1.0$. In all these figures, the very close agreement between simulation and exact expressions confirms that the choice of Monte Carlo design parameters discussed before is adequate. The values of the variances and of spectral densities at $\omega = 1$ are recorded in Tables 1 and 2, respectively. These tables help us to understand how the response of the deterministic system changes when a random parameter is present.

Random System

The value of the damping is kept constant and equal to 0.1; the decay constants α and β can assume the values (0.01, 0.1,

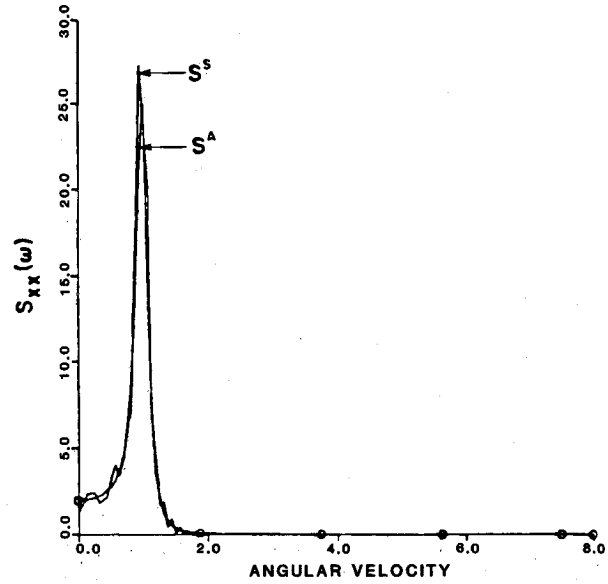

Fig. 3 Deterministic system, $\alpha = 0.1$ (S^A: analytic; S^S: simulated).

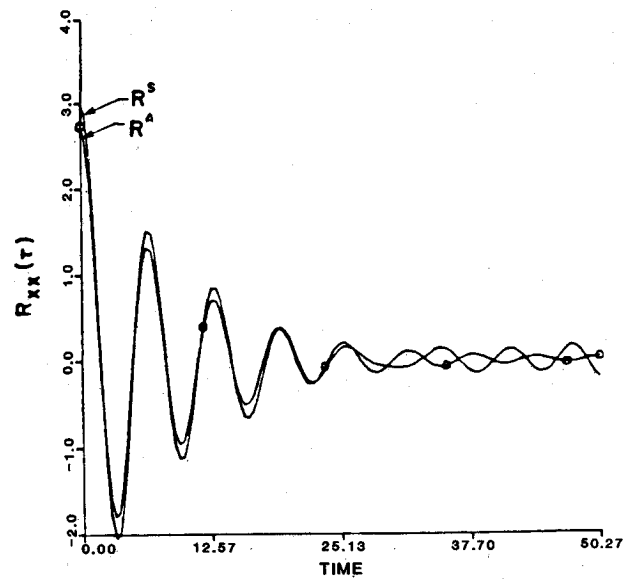

Fig. 4 Deterministic system, $\alpha = 1.0$ (R^A: analytic; R^S: simulated).

Table 1 Variance $R_{xx}(0)$

α		0.1	1.0
Deterministic system	R^s	1.5	3.0
$K = 0$	R^A	1.5	2.7
Random system	R^s	1.6	3.1
$\beta = 0.1$, $\sigma_K^2 = 0.02$	R^A	1.6	3.2
$\beta = 1.0$, $\sigma_K^2 = 0.2$	R^s	3.1	5.3
	R^A	2.3	4.3

Table 2 Peak spectral density at $\omega = 1$: $S_{xx}(1)$

α		0.1	1.0
Deterministic system	S^s	5.1	24.0–25.5
$K = 0$	S^A	5.0	25.0
Random system	S^s	4.2	22.8–29.0
$\beta = 0.1$, $\sigma_K^2 = 0.02$	S^A	6.4	31.5
$\beta = 1.0$, $\sigma_K^2 = 0.2$	S^s	11.3	29.9–45.0
	S^A	12.8	40.8

1.0, 10.0). The variance of $K(t)$, σ_K^2, is determined by trial and error. If the variance is too large for the given values of ζ, β, the response becomes unbounded. The mean value of $x(t)$ remains zero since signs alternate, but the magnitudes increase very rapidly. The variance of $x(t)$ becomes unbounded when the system is unstable in the mean square sense.[10] This phenomenon is detected numerically by partitioning the total number of simulations M into blocks of N points for which the variance is computed. The choice of block length is arbitrary, but a plot of these (M/N) blocks will reveal obvious growth patterns, thereby indicating instability. Three possible cases illustrated in Figs. 5–7 represent examples of a stable, possibly unstable, and clearly unstable system. This practical stability criterion is useful only in eliminating obviously unstable systems and fails around the stability limit. A value of σ_K^2 that leads to unstable results is tried at first; a smaller value is used next until comparisons agree. The final value of σ_K^2 is approximate and only its order of magnitude is representative of the exact stability limit.

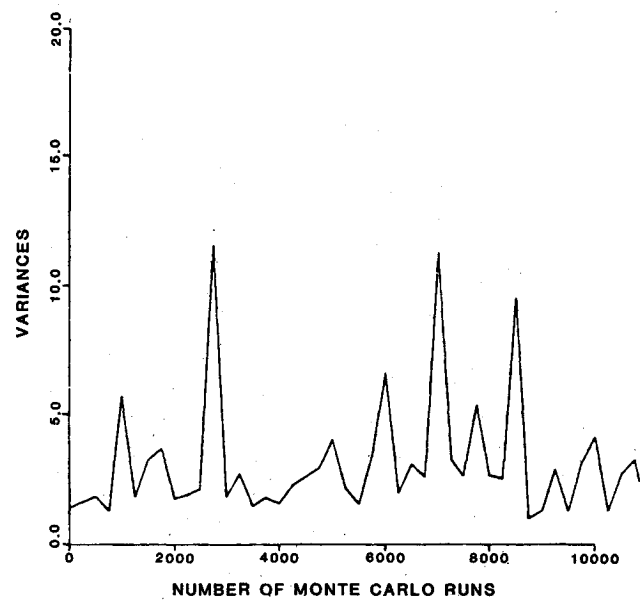

Fig. 5 Variances of blocks of 128 points. Stable system, $\zeta = 0.2$, $\sigma_K^2 = 0.4$.

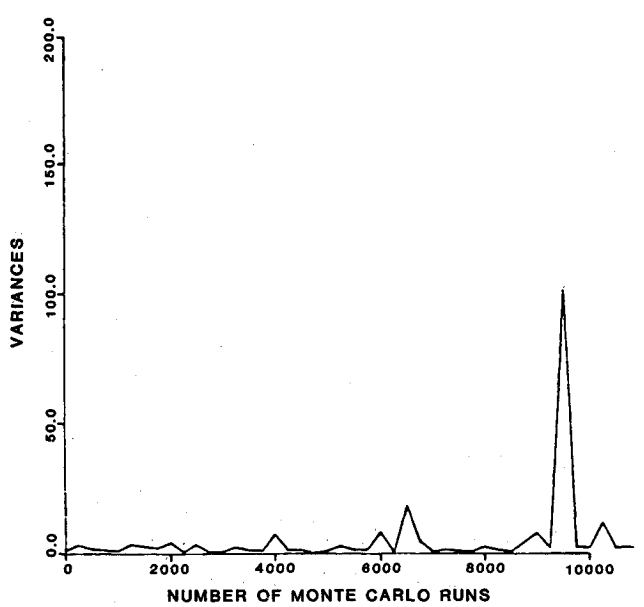

Fig. 6 Variances of blocks of 128 points. Possibly stable system, $\zeta = 0.4$, $\sigma_K^2 = 0.8$.

Extreme Values, $\beta = (0.01, 10.0)$

When $\beta = 0.01$, $K(t)$ behaves almost as a deterministic quantity since it is practically always correlated. In addition, the numerical stability criterion requires that the values of σ_K^2 be so small ($\sigma_K^2 = 0.05$) that the correction term of the iterative method is negligible with respect to the solution to the deterministic system.

When $\beta = 10.0$, $K(t)$ behaves almost as a white noise proacess since it is practically uncorrelated. The approximation is the first term only of the infinite series.[6] The solution cannot be correctly approximated by one term of the series when it is the result of the infinite summation.

It is desirable to use the exact solution that is available in both cases. Note that the approximation reduces to the deterministic solution for $\beta = 0.01$, but does not reproduce the white noise solution at the other limit when $\beta = 10.0$ since more terms of the series are needed.

Intermediate Values, $\beta = (0.1, 1.0)$

If we start with a trial variance $\sigma_K^2 = 1.0$, the autocorrelation and spectral density curves are obtained with $\beta = 0.1$. The variance plots of blocks of N points indicate that the system is clearly unstable. The variance σ_K^2 is reduced to $\sigma_K^2 = 0.05$, where the stability of the system can no longer be determined from the variance plots. Further reduction to $\sigma_K^2 = 0.02$ is needed to improve the quality of the comparisons. Figures 8 and 9 illustrate responses for $\alpha = 0.1$, and Figs. 10 and 11 correspond to $\alpha = 1.0$. Numerical values recorded in Tables 1 and 2, when compared to the deterministic system case, show a slight but consistent increase from 1.0 to 1.6. When $\alpha = 1.0$, Table 2 indicates that peaks at $\omega = 1$ have increased from 25.0, when no randomness is present, to about 30 when randomness is present with $\beta = 1$. The value of $\sigma_K^2 = 0.02$ is so small that the deterministic solution is predominant.

When $\beta = 1.0$, the largest variance that gives good agreement is $\sigma_K^2 = 0.2$. Figures 12 and 13 illustrate the $\alpha = 0.1$ case. Now, Tables 1 and 2 indicate sizable differences between the present case and the deterministic system case. When $\alpha = 1.0$, Figs. 14 and 15 confirm that the one-term iterative method follows the simulation's increase in values due to randomness in the system. Table 1 shows a twofold increase in the variance values present both in simulation and approximation. The same increase can be seen in Table 2. The approximation remains very close to the simulation, even when the random stiffness introduces considerable changes in the deterministic system.

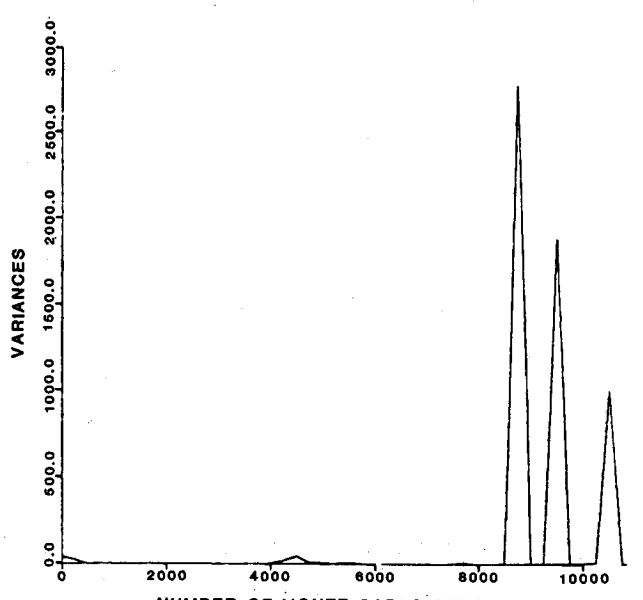

Fig. 7 Variances of blocks of 128 points. Unstable system, $\zeta = 0.8$, $\sigma_K^2 = 1.6$.

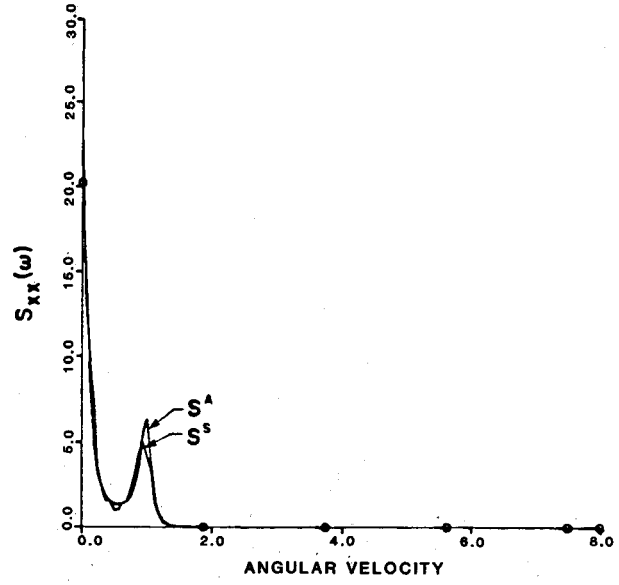

Fig. 8 Random system with $\beta = 0.1$, $\sigma_K^2 = 0.02$, and $\alpha = 0.1$ (S^A: analytic; S^S: simulated).

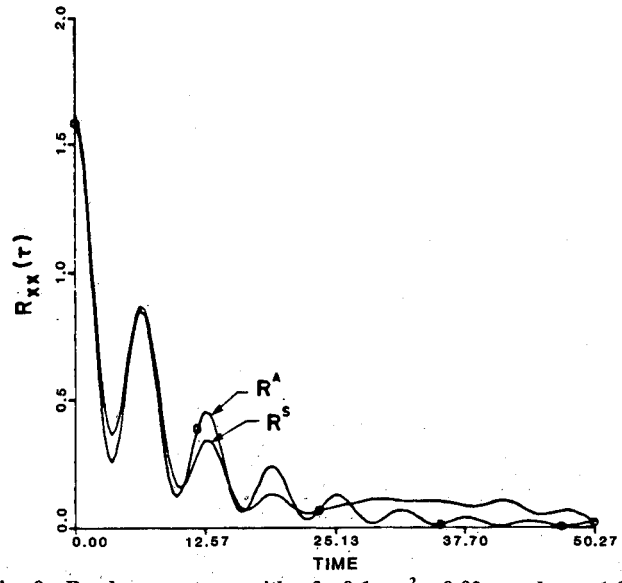

Fig. 9 Random system with $\beta = 0.1$, $\sigma_K^2 = 0.02$, and $\alpha = 1.0$ (R^A: analytic; R^S: simulated).

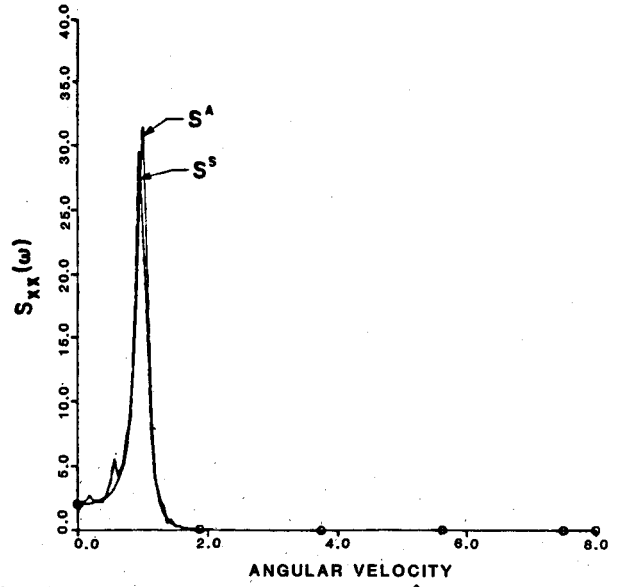

Fig. 10 Random system with $\beta = 0.1$, $\sigma_K^2 = 0.02$, and $\alpha = 1.0$ (S^A: analytic; S^S: simulated).

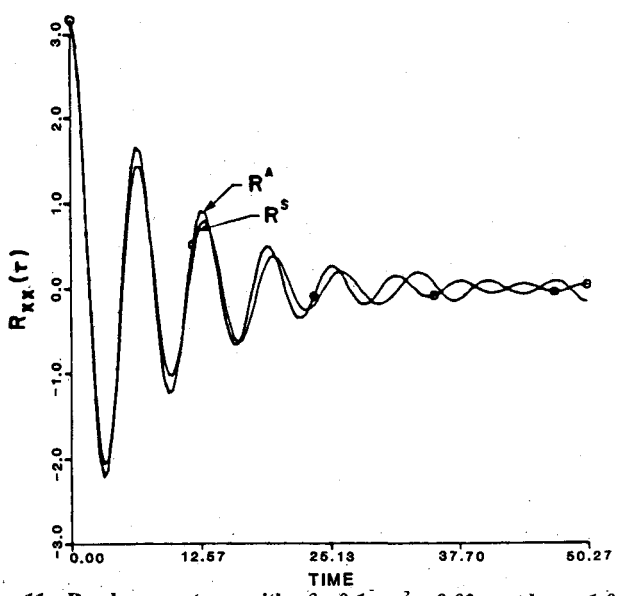

Fig. 11 Random system with $\beta = 0.1$, $\sigma_K^2 = 0.02$, and $\alpha = 1.0$ (R^A: analytic; R^S: simulated).

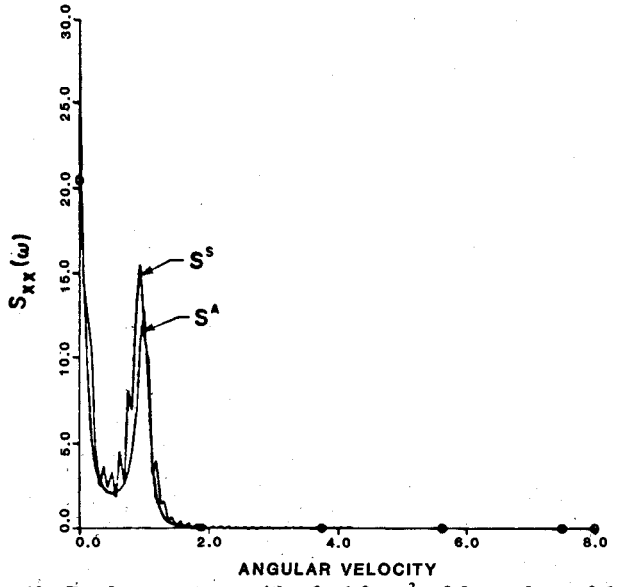

Fig. 12 Random system with $\beta = 1.0$, $\sigma_K^2 = 0.2$, and $\alpha = 0.1$ (S^A: analytic; S^S: simulated).

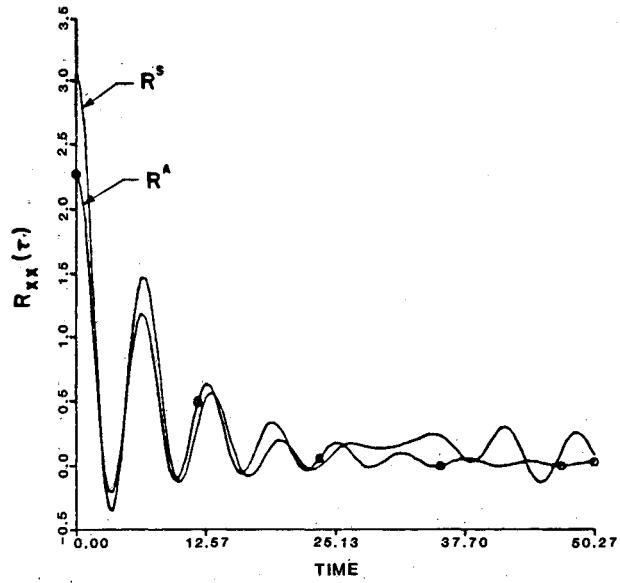

Fig. 13 Random system with $\beta = 1.0$, $\sigma_K^2 = 0.2$, and $\alpha = 0.1$ (R^A: analytic; R^S: simulated).

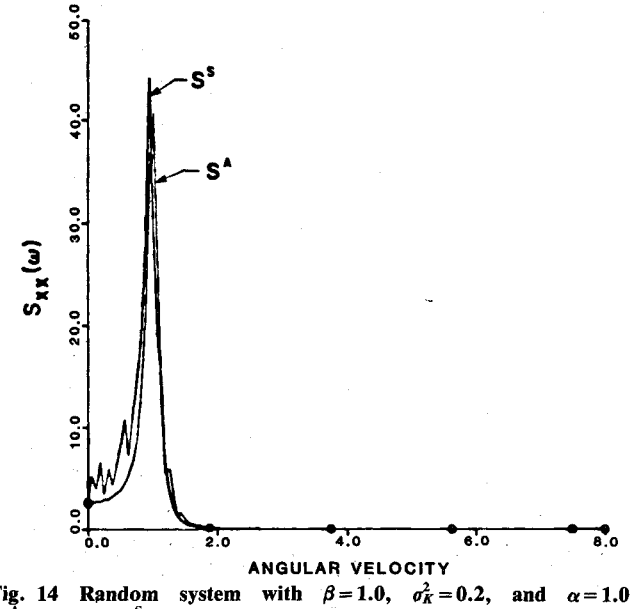

Fig. 14 Random system with $\beta=1.0$, $\sigma_K^2=0.2$, and $\alpha=1.0$ (S^A: analytic; S^S: simulated).

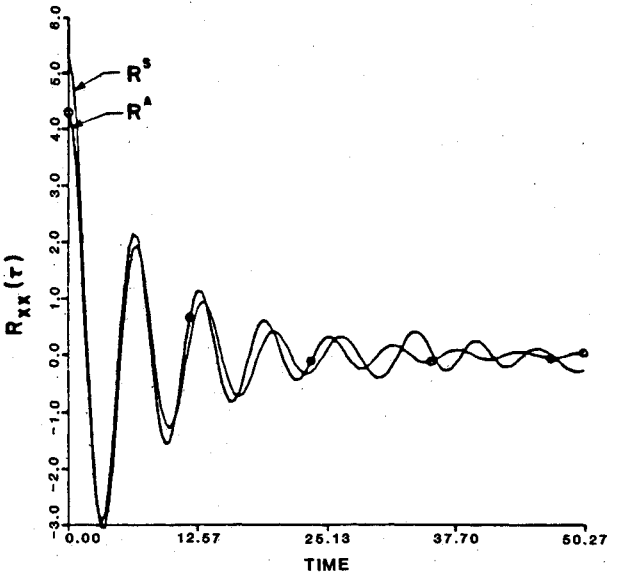

Fig. 15 Random system with $\beta=1.0$, $\sigma_K^2=0.2$, and $\alpha=1.0$ (R^A: analytic; R^S: simulated).

Criteria of Applicability

An extensive numerical study in Ref. 7, of which we have presented only a representative sample, leads to the following results:

$\beta=0.01$	0.1	1.0	10.0	
$\sigma_K^2=0.002$	0.02	0.2	white noise	

When $K(t)$ is colored, this table indicates that σ_K^2 varies with the order of magnitude of β. As a first approximation, σ_K^2 varies linearly with β. Additional information discussed in Ref. 6 regarding the dependency of σ_K^2 on ζ comes from the white noise case in which the stability limit is $S_K<4\zeta$. A white noise process of intensity S_K can be considered as a colored noise process of variance σ_K^2 and decay constant β, provided β is much larger than $\omega_0=1$, the normalized natural frequency of the system, and $\sigma_K^2=S_K(\beta/2)$. This relation is used to assign the approximate limit to $\sigma_K^2:\sigma_K^2<2\beta\zeta$ where the scaling factor $\beta/2$ has been applied to 4ζ. When $K(t)$ is colored noise, this limit is approximate and can be used only as a first guess. For the current value of $\zeta=0.1$, this limit can certainly be used as a limit within which the system is not only stable but where the approximation is valid. Most engineering structures are characterized by small values of ζ (0.01 to 0.1). In an effort to support this criterion, a parametric study on ζ reveals[7] that it fails around $\zeta=0.4$ that well covers the range of small values for ζ. No major dependency of the applicability limit on α was noted. This is expected since the stability of the system is independent from the specific forcing function.

Conclusions

The numerical simulation of a single degree-of-freedom oscillator was performed in order to establish ranges of applicability of the one-term iterative method. It was found that within numerically determined stability regions, the parameters determining the applicability of the method are the variance σ_K^2, decay constant β of the random parameter and the damping ratio ζ. The limit on σ_K^2 resulting in good agreement varies linearly with β and ζ for values of ζ around 0.1 or smaller.

The mathematical restrictions on σ_K^2 are related to the following two physical facts: 1) the system should be stable and 2) the series expansion of the iterative method should converge relatively fast so as to validate the truncation after the first correction term.

For very small decay constant β, $K(t)$ is almost deterministic and the deterministic system solution should be used. For large values of β, $K(t)$ is a white noise process for which the exact solution should also be used. For intermediate values of β, the one-term iterative method offers an improvement over the solution, neglecting randomness in the system.

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